

# *Using Graphing Calculator*

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## *Abstract*

This work presents a collection of examples of increasing difficulty that demonstrate the power of the commercial software Graphing Calculator (GC). The author developed these examples using GC in courses such as multivariable calculus and investigation of operations. The software GC and Mathematica have been fundamental to providing technological support for these courses. Such tools open the door to a new approach to the teaching and learning of mathematics.

## *1 - Introduction*

It is clear that the use of computers impacts our daily lives. Education does not escape this reality, and we need to consider seriously the incorporation of this tool into many of the disciplines that are available in Costa Rican education centers. Mathematics is no different in this regard, and today much software is available that makes it possible to approach many branches of mathematics from a different and more attractive perspective. Among the programs that are outstanding in this process, we can mention Mathematica, Maple, MuPad, Geometer's Sketch Pad and, of course, Graphing Calculator (which we discuss here).

## *2 - What is Graphing Calculator?*

GC is commercial software that allows one to visualize mathematical objects in two, three and four dimensions. It is possible to create animated graphics, to solve equations and to change the graphical point of view, among other things. GC allows one to graph functions and relations that are stated implicitly, explicitly or parametrically, in two or three dimensions.

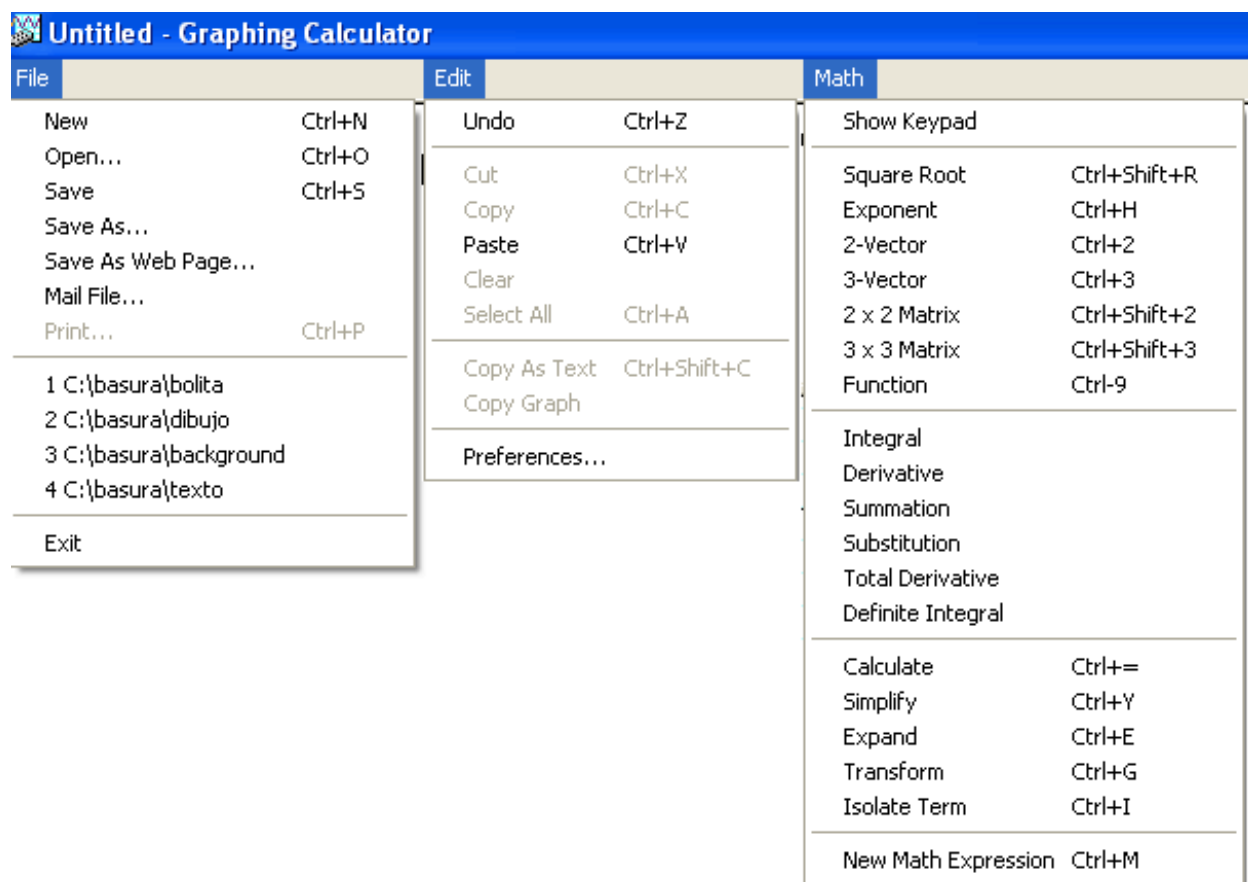
One of the notable features of this software is the simplicity with which the data are introduced. In the case of functions in two or three variables, it is not necessary to isolate a particular variable. The equation is entered as it is.

GC is available at [www.PacificT.com](http://www.PacificT.com) in a version of approximately 3 MB for Windows. In addition, a version called gcViewer can be downloaded free from this site. The free version is not 100% functional, but it allows one to visualize examples or archives developed previously by a licensed version of the program. Really, it is worth the trouble to download this version and to enjoy, in just a short time, a tool that is in the opinion of the author, virtually a digital jewel.

## *3 - Examples Using Graphing Calculator*

In this section a collection of examples of increasing difficulty appears that allow one to appreciate the facilities of GC. The following figure shows the first three menus of GC:

## Using Graphing Calculator



**Note:** In the collection of examples, expressions have sometimes been presented in a form slightly different from that which is usual in GC. This has been done to conserve space.

**Example 1** - GC can be used like calculator. Calculate a decimal approximation for the expression  $5\sqrt{3} + 2\sqrt{3} + 1$ . (The square-root button of the "keypad" can be used to facilitate the calculation.)

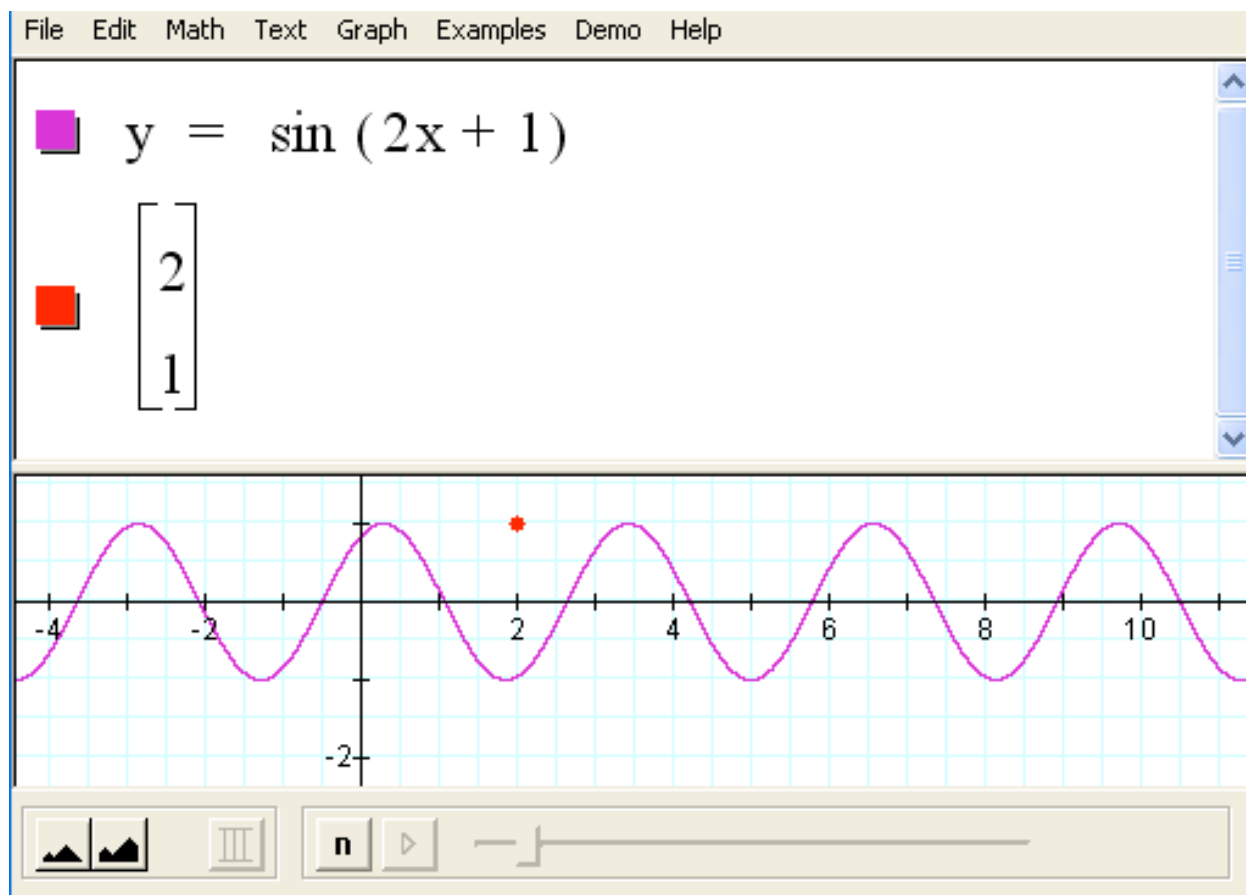
$$5\sqrt{3} + 2\sqrt{3} + 1 = 13.124356$$

Curve is outside the region shown

In GC a simple equation can be entered to be graphed.

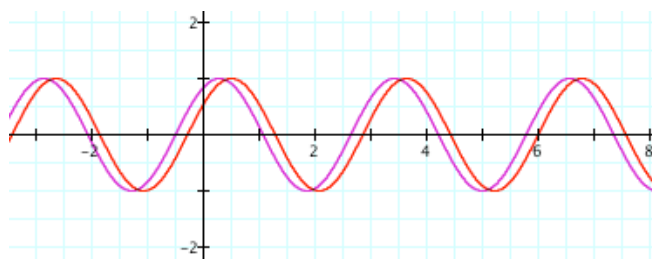
**Example 2** - In order to graph  $y = \sin(2x + 1)$  the equation is entered as written. If it is desired to mark the point (2, 1), a 2-vector is used (located in the Math menu). The result follows.

## Using Graphing Calculator



**Example 3** - Graph the two functions  $y = \sin(2x+1)$  and  $y = \cos(2x-1)$  simultaneously. (In order to add a new math expression we used the command Ctrl-M.) The result is:

■  $y = \sin(2x + 1)$   
■  $y = \cos(2x - 1)$



**Example 3** - Graph

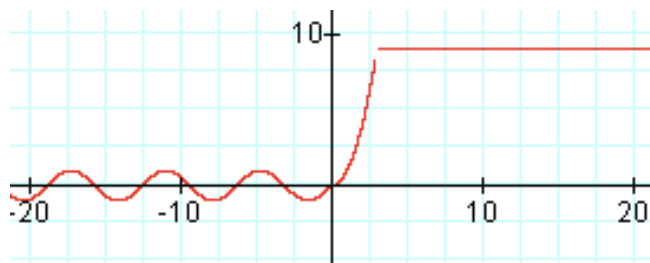
$$f(t) = \begin{cases} \sin(x) & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x < 3 \\ 9 & \text{if } x \geq 3 \end{cases}$$

## Using Graphing Calculator

Unfortunately, although in the version used here that supposedly has the symbols  $\leq$  and  $\geq$ , these never appeared. For graphical effects this means that there will be a pair of holes in the connection points. For the case of two-dimensional or three-dimensional figures the situation is corrected by separately graphing the desired region (using inequalities) and then the edge (using equations). Normally this does not represent a great problem, but you should be aware of this deficiency.

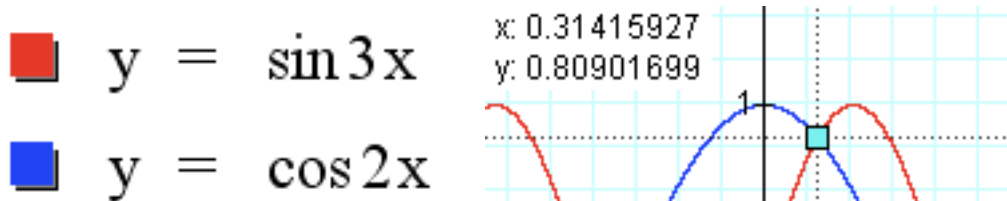
This is entered as follows:

- $y = \sin x, x < 0$
- $y = x^2, x > 0, x < 3$
- $y = 9, x > 3$



GC can be used to solve equations graphically.

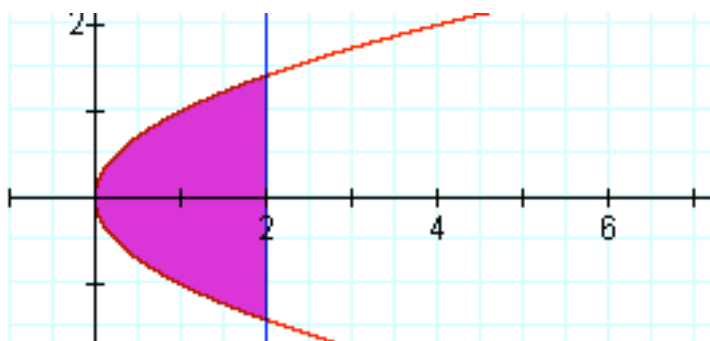
**Example 5** - Find a pair of solutions for the equation  $\sin 3x = \cos 2x$ . Using the mouse we can solve this graphically. Locate and click on the intersection of these curves to observe the numerical approximation of the  $x$  and  $y$  coordinates of the intersection. Additionally, if the computer sound is turned on, a click can be heard to when crossing an intersection of the two curves.



**Example 6** - GC can graph inequalities. Graph the region between  $x = y^2$  and straight line  $x = 2$ .


This is entered in GC is as follows:

- $x > y^2, x < 2$
- $x = y^2$
- $x = 2$

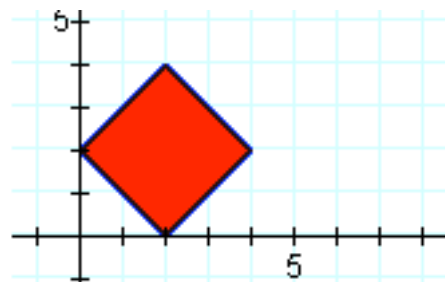


## Using Graphing Calculator

**Example 7** - Graph the rhombus given by  $|x - 2| + |y - 2| \leq 2$ . In this case we graph the interior and the edge separately. These expressions are entered in GC as it follows:


  $|x - 2| + |y - 2| = 2$


  $|x - 2| + |y - 2| < 2$

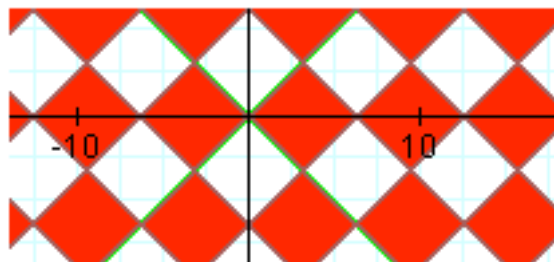


**Example 8** - Graph the region determined by  $\cos x \leq \cos y$ .

In this case we obtain a red checkerboard with green edges. We simply wrote:


  $\cos x = \cos y$

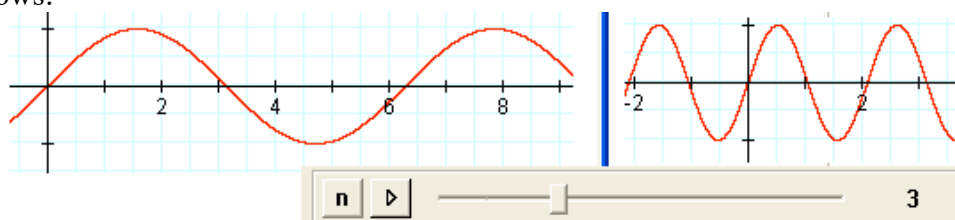
  $\cos x < \cos y$



**Example 9** - One of the most interesting aspects of GC is the animation. Graph  $y = \sin nx$ .

This is entered as follows:

  $y = \sin nx$



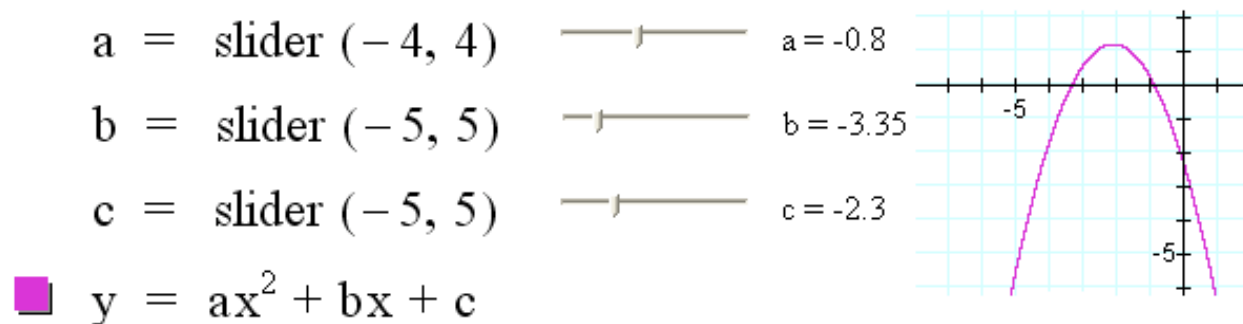
Use the scroll bar that is at the bottom of the GC window to control the animation. If you click on the button to the right of the  $n$  button, the parameter  $n$  changes automatically. To control the range of  $n$  and the speed of the animation, click on the  $n$  button itself.

**Example 10** - Graph the parabola and  $y = ax^2 + bx + c$  for  $-4 \leq a \leq 4$ ,  $-5 \leq b \leq 5$  and  $-5 \leq c \leq 5$ .

For this we must use sliders. Indicate the name of the slider and its size. Moving the control of each slider the effect of each coefficient appears immediately in this quadratic function.

This is entered in GC as follows:

## Using Graphing Calculator




**Example 11** - Graph the parametric curve:

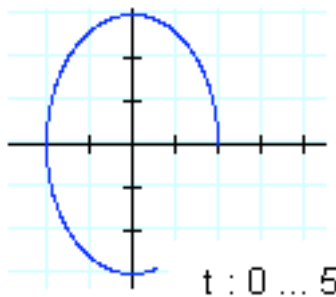
$$f(t) = \begin{cases} x = 2 \cos t \\ y = 3 \sin t \end{cases} \text{ with } 0 \leq t \leq 5$$

Observe that when changing to Cartesian coordinates we obtain  $\frac{x^2}{4} + \frac{y^2}{9} = 1$ . This is an ellipse. However, as the parameter  $t$  varies from 0 and 5, only part of this ellipse is drawn.

In order to write the vectors we use the option **2-vector** of the **Math** menu. This is entered in GC as follows:

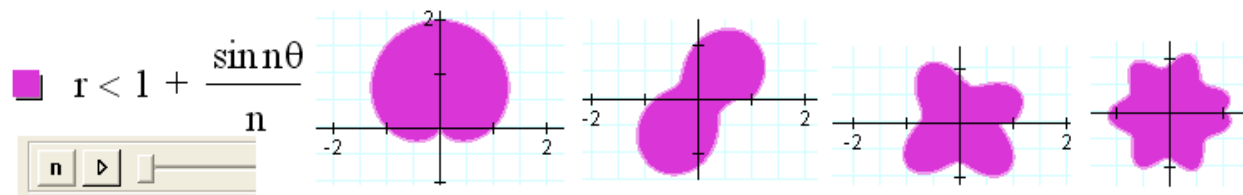


$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \cos t \\ 3 \sin t \end{bmatrix}$$



**Example 12** - Draw the graph in polar coordinates of  $r < 1 + \frac{\sin(n\theta)}{n}$  for  $n$  varying between 1 and 10.

This is entered in GC as follows:



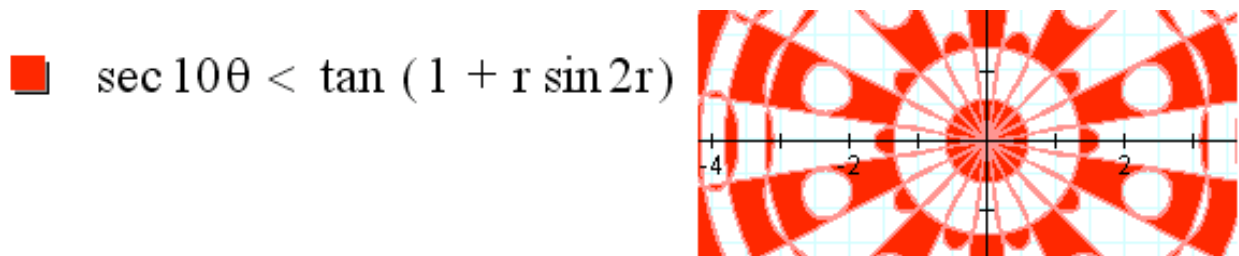
## Using Graphing Calculator

The size of  $n$  and the speed of the animation are defined in the slider bar at the base of the GC window.

In polar coordinates interesting graphics can be created. Consider the following case.

**Example 13** - Draw the graphic of  $\sec(10\theta) < \tan(1 + r \sin(2r))$ .

This is entered in GC simply as:



**Example 14** - Trace a tangent line moving along the curve of the function  $f(x) = (\pi x)$  at the point  $(n, f(n))$  as  $n$  varies.

We begin by declaring the function in question using the option **function** of the **Math** menu. In this way we can easily change to a different function. Next we graph the function by writing  $y =$

$f(x)$ . Then we declare (using the **function** and **derivative** options) the function  $g(x) = \frac{\partial}{\partial x} f(x)$ .

We know that for the point  $(n, f(n))$ , the tangent line has equation  $y = f'(n)(x - n) + f(n)$ , or  $y = g(n)(x - n) + f(n)$ . Then we graph (using **function**) this straight line: Finally we draw the point of tangency by means of the option 2-vector.

This is all entered in GC as:

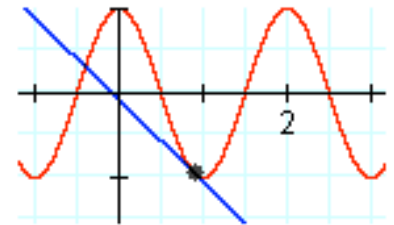
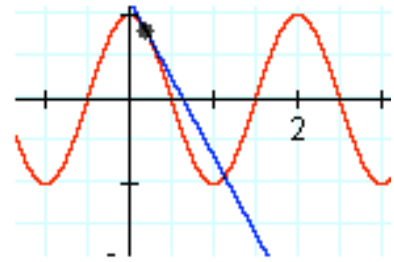
## Using Graphing Calculator

$$f(x) = \cos \pi x$$

■  $y = f(x)$   
 $g(x) = \frac{\partial}{\partial x} f(x)$

■  $y = g(n) (x - n) + f(n)$

■  $\begin{bmatrix} n \\ f(n) \end{bmatrix}$



**Example 15** - Find the coordinates of the maximum of the given function graphically over the indicated region.

$$\text{Maximize } z = 3x + 2y$$

$$\begin{cases} x + 2y \leq 6, \\ 2x + y \leq 8, \\ -x + y \leq 1, \\ y \leq 2, \\ x \geq 0, y \geq 0 \end{cases}$$

We must remember that the domain of such a linear system corresponds to a polygon. In this case we have a small detail to consider when using GC because of the appearance of the symbol  $\leq$ . As is known, the difference between  $<$  and  $\leq$  is whether or not the boundary of the inequality is included. We used GC to graph the domain excluding the border of the resulting polygon, replacing  $\leq$  by  $<$ . However, we must always determine whether the vertices of the polygon belong to the domain of the linear system. We get GC to draw the domain polygon as follows:

■  $x + 2y < 6, 2x + y < 8, -x + y < 1, y < 2, x > 0, y > 0$

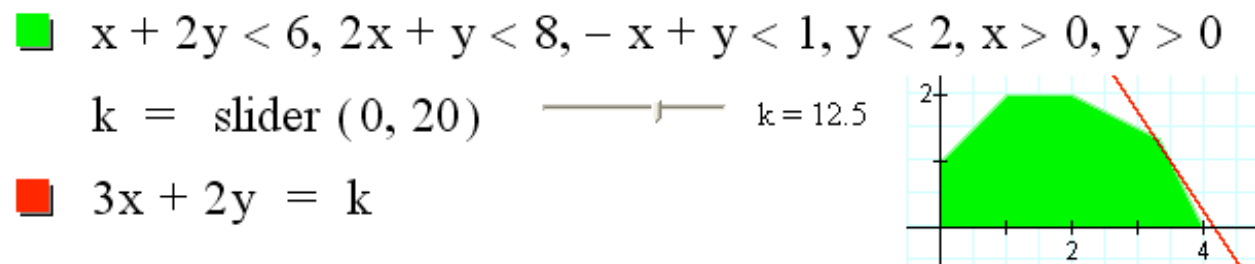
Now we have to draw the objective function. For this we declare a slider as follows  $k = \text{slider}(0, 20)$ . We then graph the straight line associated with the objective function,  $y = 3x + 2y = k$ . If we vary the slider  $k$  we notice that we exit the polygon at the vertex with approximate coordinates (3.3, 1.3). We observe that this takes place when the value of  $k$  (that corresponds to  $z$ ) is approximately 12.7. Using GCs zoom button we can obtain a better estimate for these values.



## Using Graphing Calculator

Only one more thing before we go to another example. It is likely that the reader asks why we did not declare  $z$  as the value for the slider and simply graph  $3x + 2y = z$  instead of  $3x + 2y = k$ . The problem is that GC graphs in three dimensions when it notices the presence of the variable  $z$  and in such a case graphs the plane  $3x + 2y = z$ , which is not what we want to observe.

The whole problem is entered as:



```

r[t_] = Simplify[{f[t], g[t]}, f[t] ∈ Reals ∧ g[t] ∈ Reals]
"Point at which we want to calculate the osculating circle"
s = r[p]
TT[t_] = Simplify[r'[t] / Norm[r'[t]], t ∈ Reals] //.
  Abs[y : _] ^ 2 → y ^ 2
NN[t_] = Simplify[TT'[t] / \ Sqrt[TT'[t] * TT'[t]], t ∈ Reals] //.
  Abs[y : _] ^ 2 → y ^ 2
x = Simplify[
  {∂t r[t][[1]] ∂t,t r[t][[2]] - ∂t,t r[t][[1]] ∂t r[t][[2]]} /
  Norm[∂t r[t]] ^ 3, t ∈ Reals] //. {t → p, Abs[y : _] ^ 2 → y ^ 2};
"Circle radius of curvature:"
ρ = 1 / x
"Circle:"
MC = Simplify[Norm[{x, y} - (s + ρ NN[t0])] ^ 2, x ∈ Reals ∧ y ∈ Reals] //.
  Abs[y : _] ^ 2 → y ^ 2;
MC1 = MC //. {g'[p] f''[p] - f'[p] g''[p] → α, f'[p] ^ 2 + g'[p] ^ 2 → β,
  -g'[p] f''[p] + f'[p] g''[p] → -α};
MC2 = ρ ^ 2 //. {g'[p] f''[p] - f'[p] g''[p] → α, f'[p] ^ 2 + g'[p] ^ 2 → β,
  -g'[p] f''[p] + f'[p] g''[p] → -α};
"Simplifying"
MC1 == MC2

```

Figure 1: Mathematica code for the osculating circle equation to a function at a specific point.

**Example 16** - With some effort and using Mathematica (see Figure 1) we can deduce a formula to find the equation of the osculating circle for a parameterized curve as

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$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

For the case  $t = m$ , we simplify and define:

$$j = f'(m) g''(m) - g'(m) f''(m)$$

$$k = [f'(m)]^2 + [g'(m)]^2$$

The equation of the osculating circle is then:

$$\left( y - g(m) - \frac{k f'(m)}{|j|} \right)^2 + \left( x - f(m) + \frac{k g'(m)}{|j|} \right)^2 = \frac{k^3}{j^2}$$

Note: This formula does not appear in traditional multivariable calculus books.

Consider the case of the parabola  $y = x^2$  that can be described parametrically as  $x = t$ ,  $y = t^2$ . To choose any point on the curve, we use the slider  $m$ . In addition we make  $f(t) = t$  and  $g(t) = t^2$  and we draw the curve. The code in GC is:

$$m = \text{slider}(-20, 20) \quad \text{—————} \quad m = 2.2$$

$$f(t) = t$$

$$g(t) = -t^2$$

$$\blacksquare \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

Next we define some auxiliary functions for first derivatives of  $f$  and  $g$ , respectively:

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$$\begin{array}{l|l} a(x) = \frac{\partial}{\partial x} f(x) & c(x) = \frac{\partial}{\partial x} g(x) \\ b(x) = \frac{\partial}{\partial x} a(x) & d(x) = \frac{\partial}{\partial x} c(x) \end{array}$$

Next we declare additional values  $j$  and  $k$  and finally we draw the osculating circle:

$$j = a(m)d(m) - c(m)b(m)$$

$$k = a(m)^2 + c(m)^2$$




$$\blacksquare \quad \frac{k^3}{j^2} = \left( y - g(m) - \frac{ka(m)}{|j|} \right)^2 + \left( x - f(m) + \frac{kc(m)}{|j|} \right)^2$$


GC also allows another level of interaction when it works with complex numbers of type  $z = a + bi$ , with real  $a$  and real  $b$ . As is known, an identification can be established between a complex number  $z = a + bi$  and a point in the plane  $(a, b)$ . When such a point is declared, GC allows you to change its location with the mouse and to recalculate its coordinates.


**Example 17** - Consider the point  $(1, 0)$ . An animated graphic that makes this point rotate around the origin. Draw in addition the segment that unites the points  $p = 1 + i$  and  $q = -1 - i$ .


As is known the point  $(1, 0)$  can identified with  $\cos t + i \sin t$  when  $t = 0$ , or (in exponential form) with  $e^{it}$ , when  $t = 0$ . If we let  $t$  vary between  $0$  and  $2\pi$ , we obtain the graph of the unit circle. In this form the equation for a unit circle is  $z = e^{it}$ . On the other hand if we added the equation  $z = e^{in}$  we managed to make the animation that shows the point  $(1, 0)$  rotating around the origin. Next we declare the points  $p = 1 + i$  and  $q = -1 - i$ . The equation of the straight line segment that unites  $p$  with  $q$  is  $z = (q - p)t + p$ . Nevertheless, by virtue of which we speak indicates that parameter  $t$  varies between  $0$  and  $2\pi$ , we have a good reason to introduce the modulus function. The complete code in GC is:


## Using Graphing Calculator

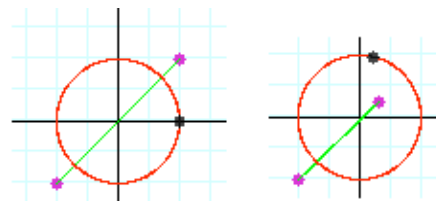
  $z = e^{2\pi it}$

  $q = 1 + i$

  $z = e^{2\pi in}$

  $p = -1 - i$

  $z = (q - p) \bmod (t, 1) + p$




When moving point  $q$  with the mouse, notice the instantaneous change of coordinates.


**Example 18** - Create an animation, in the basic system, that shows the function  $y = |x|$  rotating around the origin.

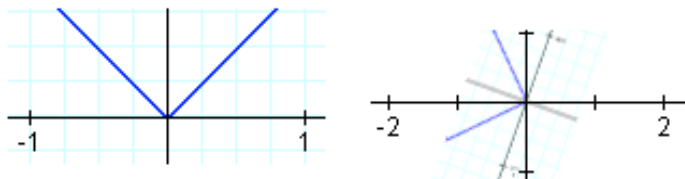
The transformation of coordinates  $z' = az$  with  $a = e^{in}$  carries out a rotation as  $n$  varies.

The code in GC is as follows:

  $z' = az$

$a = \cos n + i \sin n$

  $y = |x|$



**Example 19** - Draw a normal vector to the curve  $y = \sum_{k=1}^5 \sin^5 x$  calculating the derivative to find the slope of the tangent at  $x = n$ .

If we have a curve of the form  $y = f(x)$ , this can be parameterized as  $r(t) = \langle t, f(t) \rangle$ . Let us designate with  $A = (n, f(n))$  an arbitrary point on the curve where we want to draw the normal vector. We know that  $r'(t) = \langle 1, f'(t) \rangle$  and therefore the tangent vector is given by

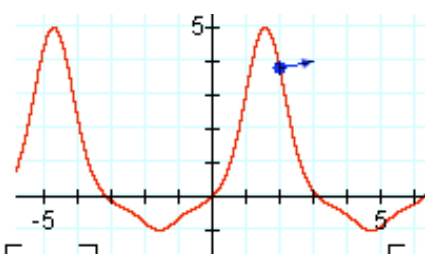
$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle 1, f'(t) \rangle}{\sqrt{1 + (f'(t))^2}}$$

The normal vector  $N(t)$  is orthogonal to  $T(t)$  and for this case (observing the concavity of the curve of interest) it can be written as

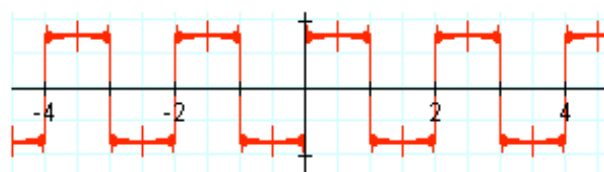
$$N(t) = \frac{\langle -f'(t), 1 \rangle}{\sqrt{1 + (f'(t))^2}}$$

## Using Graphing Calculator

In order to draw the normal vector in GC we indicated the point at which it begins and the point in which it terminates. The GC code for this problem is:

$$\begin{array}{l}
 \blacksquare \quad f(x) = \sum_{k=1}^5 \sin^k x \quad \blacksquare \quad \begin{bmatrix} n \\ f(n) \end{bmatrix} \\
 \blacksquare \quad y = f(x) \\
 \blacksquare \quad g(x) = \frac{\partial}{\partial x} f(x) \quad \blacksquare \quad \begin{bmatrix} n \\ f(n) \end{bmatrix}, \begin{bmatrix} n \\ f(n) \end{bmatrix} + \frac{1}{\sqrt{1 + g(n)^2}} \begin{bmatrix} -g(n) \\ 1 \end{bmatrix}
 \end{array}$$


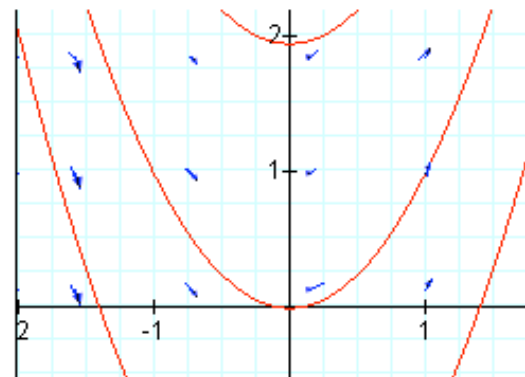
Note: The summation is entered by means of the **Summation** option of the **Math** menu. We can then construct approximations to square waves as for example:

$$\blacksquare \quad y = \sum_{k=0}^{20} \frac{\sin \pi (2k + 1) x}{2k + 1}$$


**Example 20** - Draw some solutions of the differential equation  $y' = 2x$  and its slope field.

Using the method of separation of variables for differential equations it is easy to see that the general solution of this is  $y = x^2 + k$ . Using the **Total derivative** option of the **Math** menu we

write the expression for  $\frac{dy}{dx}$ . Next we indicate the corresponding vector field that shows the slope field of this equation.

$$\begin{array}{l}
 \blacksquare \quad \frac{dy}{dx} = 2x \\
 \blacksquare \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2x \end{bmatrix}
 \end{array}$$


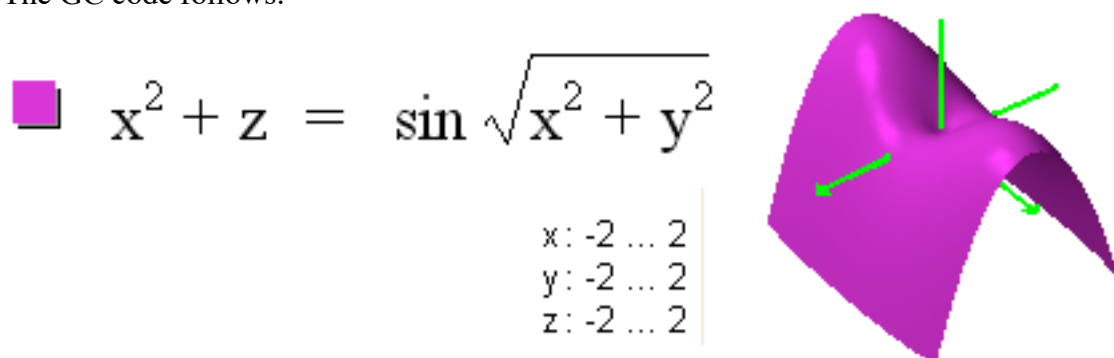
## Using Graphing Calculator

Note: Do not try to construct to the expression  $\frac{dy}{dx}$  as if it was a division (the ratio of dy to dx). You must use the **Total derivative** command from the **Math** menu.

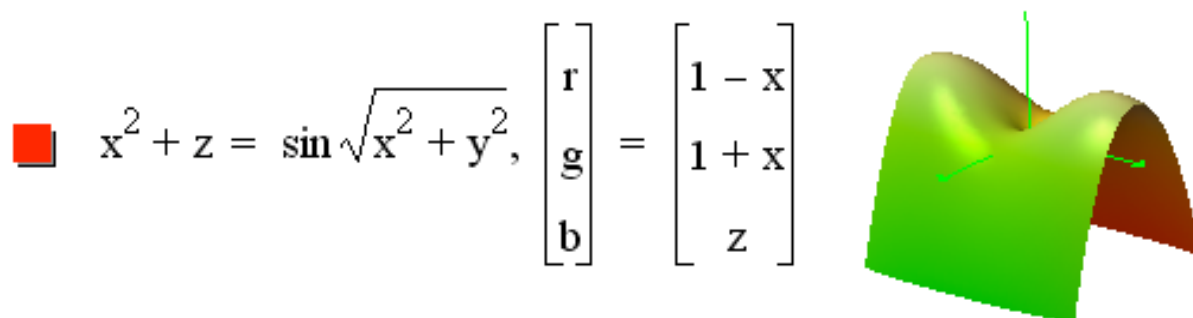
**Example 21** - Graph  $x^2 + z = \sin(\sqrt{x^2 + y^2})$ .

This is a three-dimensional surface. GC draws the graph choosing a scale for the coordinate axes and the point of view (which can be varied to observe the surface from another point of view). In fact it is possible to keep the surface rotating depending on the speed used in mouse.

The GC code follows:



Note: The z-axis is the one that does not have an arrow on its end. Once the z-axis is located, the x-axis is the one to the left and the y-axis is the one of the right. If it is desired to give a different coloring, we can use the **RGB** vector (red, green, blue) as follows:



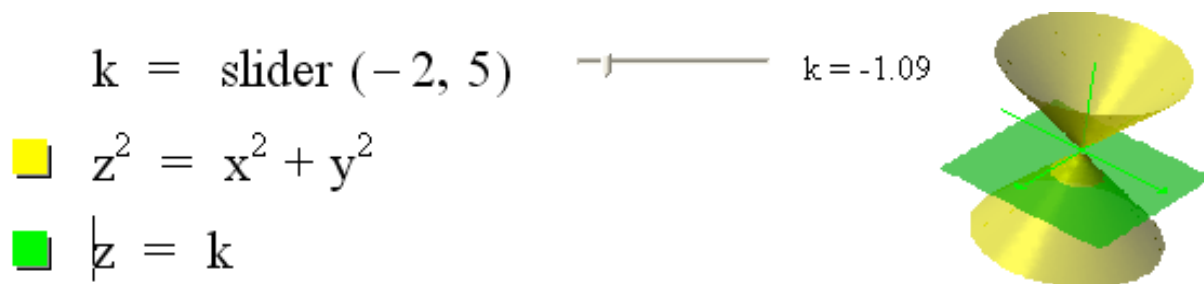
Note that each point is a function of x, y and z. Also, the color can be specified using the vector **hsv** (hue, saturation, value).

**Example 22** - Graph jointly the parabolic  $z = x^2 + y^2$  and the plane  $z = k$  for  $-2 \leq k \leq 5$ . Use transparency.

As is known  $z = k$  is a plane parallel to the xy-plane which we can move by creating a slider for

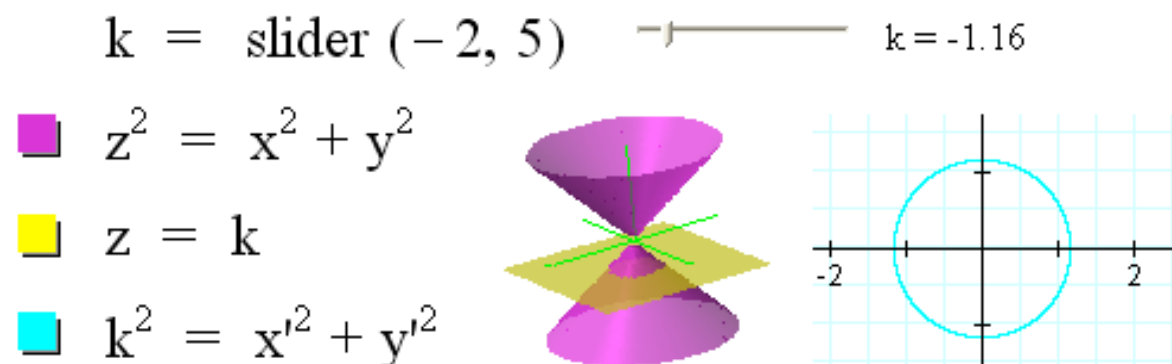
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k with the range that we specify. In the **Graph** menu we select the transparency option to see through the three-dimensional object. The GC code is:



**Example 23** - For the previous problem show adjacent graphs of the three-dimensional intersecting surfaces and the curve that one obtains when cutting the cone with the plane  $z = k$ .

The GC Code is the following:



Observe that in addition to the equation  $z^2 = x^2 + y^2$  for the cone and  $z = k$  for the plane, we add  $k^2 = x'^2 + y'^2$ . For each value of  $k$  this equation completely describes the curve that forms when cutting the cone with the plane  $z = k$ . The result is a circle of radius  $k$ . When a juxtaposed system is desired, use is made of the basic system (using the variable  $x'$  and the variable  $y'$ ).

**Example 24** - Graph the parametric curve:

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t^3 \end{cases} \quad \text{with } -5 \leq t \leq 5$$

jointly with cylinder  $x^2 + y^2 = 1$ .

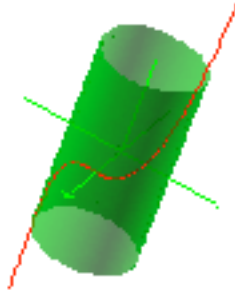
## Using Graphing Calculator

One is a three-dimensional curve. Like  $\cos^2 t + \sin^2 t = 1$ , the curve is within a cylinder and coordinate  $z = t^3$  causes it to extend throughout this cylinder. In order to construct vectors we used the option **3-vector** of the **Math** menu.

GC code is as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \\ t^3 \end{bmatrix}$$

$$x^2 + y^2 = 1$$



$x: -2 \dots 2$   
 $y: -2 \dots 2$   
 $z: -2 \dots 2$   
 $t: -5 \dots 5$

**Example 25** - Consider the following points in  $\mathbb{R}^3$ :  $A = (1, 2, 3)$ ,  $B = (1, -2, 3)$  and  $C = (-1, 2, 3)$ .

Draw the vector that goes from A to B, the one that goes from A to C and the one that goes from

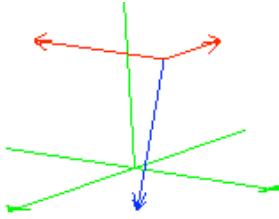
A to  $\frac{(B - A) \times (C - A)}{\|(B - A) \times (C - A)\|}$ . We use option **3-vector** of the **Math** menu. The code in GC is:

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \quad C = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

■ A, B

■ A, C

■ A,  $\frac{B - A \times C - A}{|B - A \times C - A|}$



**Example 26** Use spherical coordinates to graph the right side of the paraboloid whose equation

this given by  $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$ .


The parameterization in spherical coordinates for this case can be written as



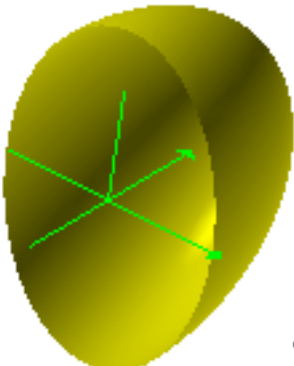
## Using Graphing Calculator

$$\begin{cases} x = 2 \sin u \cos v \\ y = 4 \sin u \sin v \\ z = 3 \cos u \end{cases} \quad \text{with } 0 \leq u, v \leq \pi$$

The code is the following:



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \sin u \cdot \cos v \\ 4 \sin u \cdot \sin v \\ 3 \cos u \end{bmatrix}$$



x: -2 ... 2

y: -2 ... 2

z: -2 ... 2

u: 0 ... 3.14

v: 0 ... 3.141

**Example 27** - Graph the tangent plane to the ellipsoid  $\frac{3}{4}x^2 + 3y^2 + z^2 = 12$  at the point  $(2, 1, \sqrt{6})$ . We express the first equation of the surface  $f(x, y, z) = 0$  defining

$$f(x, y, z) = \frac{3}{4}x^2 + 3y^2 + z^2 - 12 = 0.$$

We begin by declaring function **f** using the option **function** of the **Math** menu. We observe that the partial derivatives of **f** are  $f_x(x, y, z) = \frac{3}{2}x$ ,  $f_y(x, y, z) = 6y$ ,  $f_z(x, y, z) = 2z$ , and therefore at  $(2, 1, \sqrt{6})$ ,

$$f_x(2, 1, \sqrt{6}) = 3, f_y(2, 1, \sqrt{6}) = 6, f_z(2, 1, \sqrt{6}) = 2\sqrt{6}$$

In order to obtain this in GC we define some auxiliary functions (using the options **function** and **derivative** of the **Math** menu), that is:

$$g(x, y, z) = \frac{\partial}{\partial x} f(x, y, z), \quad h(x, y, z) = \frac{\partial}{\partial y} f(x, y, z), \quad k(x, y, z) = \frac{\partial}{\partial z} f(x, y, z)$$

Next we indicate the tangent point with the values a, b and c as follows:

## Using Graphing Calculator

$$a = 2 \quad b = 1 \quad c = \sqrt{6}$$

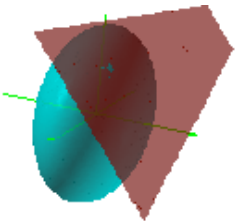
Observe that to change the tangent point, we simply modify this code. For this problem, the equation is:

$$3(x - 2) + 6(y - 1) + 2\sqrt{6}(z - \sqrt{6}) = 0$$

In GC we write this as:

$$g(a, b, c)(x - a) + h(a, b, c)(y - b) + k(a, b, c)(z - c) = 0$$

To graph the ellipsoid we simply add the code  $f(x, y, z) = 0$ . The complete code is as follows:

$f(x, y, z) = \frac{3}{4}x^2 + 3y^2 + z^2 - 12$ $g(x, y, z) = \frac{\partial}{\partial x}f(x, y, z)$ $h(x, y, z) = \frac{\partial}{\partial y}f(x, y, z)$ $k(x, y, z) = \frac{\partial}{\partial z}f(x, y, z)$	<div style="display: flex; align-items: center;"> <div style="margin-right: 20px;"> <math>a = 2</math>  <math>b = 1</math>  <math>c = \sqrt{6}</math> </div>  </div> <div style="margin-top: 10px;"> <span style="color: green;">■</span> <math>g(a, b, c)(x - a) + h(a, b, c)(y - b) + k(a, b, c)(z - c) = 0</math> </div> <div style="margin-top: 10px;"> <span style="color: magenta;">■</span> <math>0 = f(x, y, z)</math> </div>
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**Example 28** - Graph the tangent plane to the ellipsoid of the previous example

$$\frac{3}{4}x^2 + 3y^2 + z^2 = 12 \text{ at any point.}$$

In order to do this we can parameterize an arbitrary point of the ellipsoid and use its parameters as sliders. Using spherical coordinates, we have:

$$a = 4 \cos(p) \sin(q), \quad b = 2 \sin(p) \sin(q), \quad c = \sqrt{12} \cos(q)$$

We then declare p and q as sliders:

$$p = \text{slider}(0, 6.28), \quad q = \text{slider}(0, 3.14).$$

The complete code is then: The problem also can be approached using formulas for the tangent plane for parameterized surfaces. Do this as an exercise.

## Using Graphing Calculator

**Example 29** - Find the equation of the surface of revolution that is obtained when the graph of  $y = x^2$  is rotated around x-axis. Graph this surface and create an animation that shows this curve turning around the x-axis.

To find the equation of a surface revolution that is obtained when rotating a curve with equation  $F(x, y) = 0$  around the x-axis, we must replace the variable y by the expression  $\sqrt{y^2 + z^2}$ . We thus find the desired surface has equation  $\sqrt{y^2 + z^2} = x^2$ . To make the animation of the curve  $y = x^2$  rotating around x-axis we use parameterization:

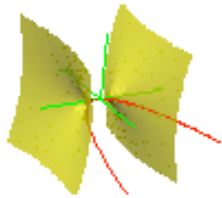
$$\begin{cases} x = t \\ y = t^2 \end{cases}$$

Note: We cannot use the cartesian form  $y = x^2$  for this animation because an equation of type  $y = f(x)$  is drawn as a cylinder when we are in working in 3D.

In order to make the curve in question rotate we use the matrix of rotation

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi n) & \sin(\pi n) \\ 0 & -\sin(\pi n) & \cos(\pi n) \end{bmatrix}$$

In order to enter this matrix in GC we make use of option **3 x 3 Matrix** of the **Math** menu. It remains to graph the parametric curve and vary parameter n. It is good to set the number of passes to 50 or less to appreciate the revolutions of the curve. The code in GC is:


$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \pi n & \sin \pi n \\ 0 & -\sin \pi n & \cos \pi n \end{bmatrix} \quad \left| \quad \begin{array}{l} \text{■} \quad \sqrt{y^2 + z^2} = x^2 \\ \text{■} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R \begin{bmatrix} t \\ t^2 \\ 0 \end{bmatrix} \end{array} \right. \quad \text{■}$$


In order to finish it is good to show an example in which GC does not shine very well, that is to say, a case in which the drawing made is not very clear.

## Using Graphing Calculator

**Example 30** - Draw the graph of  $\sin(x^2 + y^2 + z^2) = \frac{1}{n}$  for n between 1 and 10. Use vector RGB.

The code in GC is:

$$\sin(x^2 + y^2 + z^2) = \frac{1}{n}, \quad \begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} 1 - x \\ 1 + x \\ z \end{bmatrix}$$


It is a succession of concentric spheres.

### 4 - Conclusion

In the process of teaching and learning mathematics today one counts on valuable digital allies that we must learn to use and to apply. The possibility that the student, properly oriented, may discover interesting results on his own, opens the door to learning in a pleasant and beneficial way.

This work has demonstrated, by means of examples, the benefits of the tool GC and will have achieved its goal if it inspires the reader to learn more about this tool. The author is willing to collaborate in this process with interested readers.

### References

[1] T. Erickson, Learning Math with Graphing Calculator, Investigating Symbols, Functions, and Graphs, Pacific Tech, the USA, 2002.